OPTIMAL ATTACK AGAINST AN AREA DEFENSE
PROTECTING MANY TARGETS

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June 1989

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ABSTRACT

Many separated targets, of only a few differing values, are subject to a simultaneous attack. The area defenses considered have (a) impact point prediction (IPP) and full coordination, or (b) no IPP and full coordination, or (c) no IPP and partial coordination. For a given attack, the defense wishes to allocate its interceptors to maximize the expected total survival value of the targets. For a given attack size, and with knowledge of the defense's capabilities, the offense seeks a strategy to minimize expected total survival value against best defense. We present algorithms to determine optimal attack and defense strategies and the optimal value of the min-max problem, and we show how to take computational advantage of the relatively few unique target values. Illustrative computational results are provided.

1. INTRODUCTION

Consider an area to be defended that consists of T separated point targets, and associate with target i a value v_i, $i = 1, \ldots, T$. Without loss of generality, we assume that the v_i satisfy $v_1 \geq v_2 \geq \cdots \geq v_T > 0$. We denote the number of attackers directed against target i by a_i and the attack against the T targets by $a = [a_1, \ldots, a_T]$, with $\sum_{i=1}^{T} a_i = A$ and each $a_i \in \mathcal{N}$, the set of nonnegative integers. The defense has D interceptors, each of which may be assigned to intercept any attacker in a given subset, possibly the entire set, of attackers. The objective of the defense is to allocate its interceptors against attackers so as to maximize expected total survival value. In his first move, the attacker seeks an attack allocation which will minimize expected total survival value against best defense.

The specific assumptions we make are as follows:

(1) The attack is simultaneous, i.e., the defense sees the overall size of the attack, A.

(2) The defense has the "last move." Only after the attack is observed does the defense need to allocate its D interceptors.

(3) All A attackers are identical. An attacker will penetrate to its intended target if not successfully intercepted. Such a penetrator will destroy the target with probability α, $\alpha \in (0, 1]$. If multiple attackers penetrate to a target, they act independently.

(4) All D defenders (interceptors) are identical. An interceptor will destroy the attacker at which it is directed with probability δ, $\delta \in (0, 1]$. If several defenders are aimed at a single attacker, they act independently.

(5) Once an attack or defense is allocated, there is no adjustment or modification. For example, attackers may not be diverted from their initial destinations nor may the defense use a "shoot-look-shoot" option.

(6) The attacker knows the overall size of the defense, D.

(7) The adversaries have common knowledge of the target values.

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(8) If the defense has *impact point prediction* (IPP), it correctly identifies the intended target of each attacker.

(9) If the defense is *(fully) coordinated*, its resources are allocated by a single controller. If different subsets of the interceptors are allocated by independent controllers, we say the defense is *partially coordinated*.

(10) The defense has area defenders only; none of the defensive interceptors are point defenders assigned to protect specific targets.

A brief outline of the paper is as follows. In the remainder of this introductory section we briefly survey the literature dealing with problems of attack and defense of a set of separated point targets. In section 2, we introduce our problem functions and formulate the defender's problem and the attacker's problem associated with each of the three defense capabilities considered: (1) the defense has IPP and is coordinated; (2) the defense has no IPP but is coordinated; and (3) the defense has no IPP and is only partially coordinated. For each of the three cases, we provide (or reference) appropriate algorithms to solve both the defender's problem and the attacker's problem and provide the optimal min-max value. In section 3 we show how the efficiency of each of the three algorithms, for the respective versions of the attacker's problem, can be greatly increased by taking advantage of the relatively few distinct values among the T target values. Section 4 provides illustrative computational results, and section 5 contains concluding remarks.

A good deal of missile allocation work deals with preallocated preferential defense of a set of identically valued targets, using a game theoretic approach by both attacker and defender. Early work on this problem was by Matheson (1966, 1967) and recently Bracken, Brooks and Falk (1987) and Bracken, Falk and Tai (1987) have examined robustness issues connected with the assumption that the defender knows the attack size A. They provide references to the intervening work in this area.

In their seminal work, Eckler and Burr (1972) examined a number of missile allocation

problems under a variety of assumptions and thoroughly surveyed the literature.

For targets of different value, Karr (1981) analyzed several problems under the assumption of sequential attack (the defense sees the attackers one at a time and must make an engagement decision at that time without knowledge of the overall attack size) and developed continuous (i.e., noninteger) solutions on the basis of Prim-Read deployments.

There is a body of work in which the defender's objective is to minimize the cost of the defense while insuring prescribed levels of survival value as a function of attack size. Under the assumption of a linear bounding function, Burr, Falk and Karr (1985) generated integer solutions for a defense against a sequential attack. Falk (1985b) extended this concept to a larger class of bounding functions, and McGarvey (1987) added an overview with mixed strategies. Karr (1982) and Soland (1987a) developed solutions for defenses against simultaneous attacks.

Finn (1986) examined the value of the additional knowledge provided by IPP for the case of perfect weapons (i.e., $\alpha = \delta = 1$) with an area defense and terminal defenses at the targets.

2. PROBLEM FORMULATIONS AND SOLUTION ALGORITHMS

2.1 Coordinated Defense with IPP

2.1.1 Problem Formulation

If d_i interceptors are allocated for the defense of target i under an attack of size a_i, then an even-as-possible distribution of these d_i defenders against the a_i attackers is optimal. [That such a quasi-uniform defense of target i is optimal is shown by Soland (1987b)]. Specifically, for $a_i > 0$, the defense allocates

$\lceil d_i / a_i \rceil$ interceptors to each of $r_i \doteq \lceil d_i - \lfloor d_i / a_i \rfloor a_i \rceil$ attackers

and

$k_i \doteq \lfloor d_i / a_i \rfloor$ interceptors to each of $a_i - r_i$ attackers.

[Notation: We use $\lfloor x \rfloor$ for the integer part of x and $\lceil x \rceil \doteq \min \{n \geq x \mid n \text{ integer}\}$.]

If we define the probability that a target survives an attacker facing m defenders as

$$\pi_m \doteq 1 - \alpha(1-\delta)^m \text{ for } m = 0, 1, \ldots ,$$

then, with k_i and r_i as defined above, we may write the survival probability at target i as

$$\phi(a_i, d_i) = \begin{cases} \pi_{k_i}^{a_i-r_i} \pi_{k_i+1}^{r_i}, & a_i = 1, 2, \ldots, A \\ 1, & a_i = 0. \end{cases}$$

Let $d = [d_1, \ldots, d_T]$. We now define the defender's problem for a given attack a as

$$\mathcal{P}_1(a) \qquad \text{Maximize} \qquad \Phi(a, d) = \sum_{i=1}^{T} v_i \phi(a_i, d_i)$$

subject to

$$\sum_{i=1}^{T} d_i \leq D$$

$$d_i \ \varepsilon \ \mathcal{N}, \qquad i = 1, \ldots, T.$$

We denote the optimal value of problem $\mathcal{P}_1(a)$ by $v[\mathcal{P}_1(a)]$. [Note: For perfect weapons ($\alpha = \delta = 1$), the defender's problem is a 0-1 knapsack problem. Otherwise, it is a nonlinear integer programming problem.] We may then formulate the attacker's problem as

$$\mathcal{P}_1 \qquad \text{Minimize} \qquad v[\mathcal{P}_1(a)]$$

subject to

$$\sum_{i=1}^{T} a_i \leq A$$

$$a_i \ \varepsilon \ \mathcal{N}, \qquad i = 1, \ldots, T.$$

Inasmuch as our solution algorithm for problem $\mathcal{P}_1$ (itself a nonlinear integer programming problem) is of the branch-and-bound type, we would greatly prefer to limit the number of attacks that we need to examine. The following *Nonincreasing Attack Theorem* (NAT) enables us to do so. The NAT for the case of IPP and perfect weapons was first proved by Falk (1985a), whereas a proof with IPP and imperfect weapons (i.e., $\alpha < 1$ and $\delta < 1$) was suggested by Falk (1987) and given by O'Meara (1988). The proof for the case of no IPP and coordinated defense was given by O'Meara and Soland (1988b).

Theorem (Nonincreasing Attack Theorem)

There is an optimal attack, say a^*, with $a_i^* \geq a_{i+1}^*$, $\imath = 1, ..., T - 1$. We call a^* a *nonincreasing* attack.

2.1.2 Algorithm for Problem $\mathcal{P}_1$

The algorithm we use to solve problem $\mathcal{P}_1$ is a branch-and-bound scheme. During the course of the algorithm, it is often necessary to determine the "best defense" against some given attack a, that is, to solve problem $\mathcal{P}_1(a)$ (or to determine some bounds on its optimal value). Details of the algorithm are provided by O'Meara and Soland (1988a, 1988c).

2.2 Coordinated Defense with no IPP

2.2.1 Problem Formulation

To avoid any cumbersome notation, we shall examine the cases of perfect weapons ($\alpha = 1, \delta = 1$) and imperfect weapons ($\alpha < 1$, $\delta < 1$) separately. Results for the mixed cases follow without difficulty.

From the point of view of the attack, with perfect weapons there is a logical constraint of allocating no more than D + 1 attackers against any target since that number will assure target destruction. The assumption of no IPP means that the defense has no real decision since all the attackers "look the same." Accordingly, the defense allocates its D defenders against D of the attackers. Since the weapons are perfect, (A − D) attackers will penetrate to their intended targets. For nontrivial problems we have $D+1 \leq A < T(D+1)$.

We observe that target i survives only if all a_i attackers are intercepted. To determine this probability, we consider the D attackers to be intercepted as a sample drawn at random (without replacement) from the population of A attackers. If we think of this as a population dichotomized into attackers destined for target i and those that are not, a hypergeometric distribution is appropriate.

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Accordingly, we define the *survival probability at target ı against attack* $a_ı$ as $\psi(a_ı)$ and compute it as

$$\psi(a_i) = \begin{cases} \dfrac{\dbinom{a_i}{a_i}\dbinom{A-a_i}{D-a_i}}{\dbinom{A}{D}}, & a_i = 0, 1, \ldots, D \\ \\ 0, & a_i > D. \end{cases}$$

We then compute the *expected survival value at target ı* as

$$v_i\psi(a_i)$$

and the expected total survival value for attack **a** as

$$\Psi(\mathbf{a}) = \sum_{i=1}^{T} v_i\psi(a_i).$$

With imperfect weapons, any number of attackers may be directed against any target, and any number of interceptors may be allocated against any attacker. Of course, the overall attack and defense size constraints are still active.

As above, *the lack of impact point prediction means that the attackers appear identical to the* defense. Intuition suggests that the best defense is to spread the interceptors as evenly as possible among the attackers. This has been shown by Soland (1987b) to be an optimal defense in that it stochastically minimizes the number of attackers that penetrate the defense. Specifically, the defense allocates

$\lceil D/A \rceil$ interceptors to each of $R \doteq \lceil D - \lfloor D/A \rfloor A \rceil$ attackers

and

$\lfloor D/A \rfloor$ interceptors to each of $A - R$ attackers.

With system reliabilities $\alpha, \delta \in (0, 1)$, the parameter range for nontrivial problems is $A > 0$ and $D \geq 0$.

In this case the uncertainty about survival value can be attributed to both system (un)reliability and the lack of IPP. As might be expected, the survival probability function has a more

involved structure. For target i not under attack, we have $\psi(a_i) = 1$. For $a_i > 0$, conditioning arguments [see O'Meara and Soland (1988b)] yield the following:

(i) if $D < A$,

$$\psi(a_i) = \sum_{l=\underline{n}}^{a_i} (1-\alpha)^l \sum_{n=\underline{n}}^{\bar{n}} \binom{a_i - n}{l - n} (1-\delta)^{l-n} \delta^{a_i - l} \frac{\binom{a_i}{n}\binom{A - a_i}{A - D - n}}{\binom{A}{A - D}},$$

where $\underline{n} = \max\{a_i - D, 0\}$ and $\bar{n} = \min\{l, A - D\}$;

(ii) if $D \geq A$,

$$\psi(a_i) = \sum_{l=\underline{n}}^{a_i} (1-\alpha)^l$$

$$\cdot \sum_{m=0}^{l} \sum_{n=\underline{n}}^{\bar{n}} \binom{n}{m} \left[(1-\delta)^{\lfloor D/A \rfloor}\right]^m \left[1 - (1-\delta)^{\lfloor D/A \rfloor}\right]^{n-m}$$

$$\cdot \binom{a_i - n}{l - m} \left[(1-\delta)^{\lceil D/A \rceil}\right]^{l-m} \left[1 - (1-\delta)^{\lceil D/A \rceil}\right]^{a_i - n - (l-m)}$$

$$\cdot \frac{\binom{a_i}{n}\binom{A - a_i}{A - R - n}}{\binom{A}{A - R}},$$

where $\underline{n} = \max\{a_i - R, 0\}$ and $\bar{n} = \min\{a_i - (l - m), A - R\}$.

We formally define the attacker's problem as

$$\mathcal{P}_2 \qquad \text{Minimize} \qquad \Psi(\mathbf{a}) = \sum_{i=1}^{T} v_i \psi(a_i)$$

$$\text{subject to}$$

$$\sum_{i=1}^{T} a_i \leq A$$

$$a_i \in \mathcal{N}, \qquad i = 1, ..., T.$$

While it is fairly straightforward to show that, for perfect weapons, problem $\mathcal{P}_2$ is a convex programming problem in the integer sense that the function ψ has nondecreasing increments, the complexity of ψ for imperfect weapons has not yet permitted a similar theoretical assertion. However, all computational experience thus far has not uncovered a nonconvex case. The algorithm we use (see below) is of the "reduction" type for convex problems. A dynamic programming algorithm is also available for any nonconvex instances, if necessary. [See O'Meara and Soland (1988b)].

2.2.2 Algorithm for Problem $\mathcal{P}_2$ (ψ Convex)

Lafon and Lahrichi (1988) describe the following algorithm for a class of problems containing problem $\mathcal{P}_2$ when ψ has nondecreasing increments, that is, when

$$v(a_i+1) - v(a_i) \geq v(a_i) - v(a_i-1), \quad a_i = 1, 2, ..., A.$$

Algorithm [Lafon and Lahrichi (1988)]:

Step 0 (Initialization):

$a_i = 0, i=1, ..., T; \quad A_1 = A;$

$k=1; \quad$ go to step k.

Step k (Allocation):

If $A_k = 0$, stop (a is optimal for problem $\mathcal{P}_2$);

else $x_k = \lceil A_k/T \rceil$;

$i^* = \min\left\{ i \mid v_i[v(a_i+x_k) - v(a_i)] = \min_{j=1,...,T}\left\{ v_j[v(a_j+x_k) - v(a_j)]\right\}, i=1, ..., T\right\};$

$a_{i^*} := a_{i^*} + x_k;$

$A_{k+1} := A_{k+1} - x_k;$

$k := k+1; \quad$ go to step k.

Observe that the allocation $\mathbf{a} = [a_1, ..., a_T]$ at the start of any step $k \geq 1$ solves

$$\text{Minimize } \sum_{i=1}^{T} v_i \, \psi(a_i)$$

subject to

$$\sum_{i=1}^{T} a_i = A - A_k$$

$$a_i \; \varepsilon \; \mathcal{N}, \; i = 1, ..., T.$$

2.3 Partially Coordinated Defense with no IPP

2.3.1 Problem Formulation

For the case of partial coordination and no IPP, suppose the D interceptors are divided into N (*battle*) *groups* of n interceptors each, so Nn = D. Suppose each group of n interceptors is fully coordinated in the sense that it is centrally controlled. We assume, however, that different groups act independently and do not communicate with each other.

Suppose each of the N groups is physically able to engage each of m different attackers with each of its n interceptors, where $1 \leq m \leq A$.

Assume, finally, that each attacker may be defended against by r different groups, and thus by a total of nr interceptors. Hence, there are Anr possible pairings of attackers and interceptors. This number is also equal to mD, since each interceptor can be allocated to any one of m different attackers. Hence Anr = mD, so n/m = D/rA. For example, A=D=1000, n=10 and m=20 lead to r=2.

Now consider an arbitrary one of the a_i attackers directed at target i, and consider one of the r groups of interceptors able to fire at this attacker. Since the defense lacks IPP, the controller of this group can do no better than to spread his n interceptors as uniformly as possible among the m attackers he can fire at; this stochastically minimizes [see Soland (1987b)] the number of attackers (out of the m) that, in the eyes of this controller, penetrate his defense. Thus the controller selects $n - m\lfloor n/m \rfloor$ of the m attackers to receive $\lceil n/m \rceil = \lceil \eta \rceil$ interceptors each, where $\eta \doteq n/m = D/rA$, and

$m - n + m\lfloor n/m \rfloor$ of the m attackers to receive $\lfloor \eta \rfloor$ interceptors each. Our particular attacker there-fore survives the defense of this group with probability

$$\rho \doteq (1 - \delta)^{\lceil \eta \rceil}(n - m\lfloor n/m \rfloor)/m + (1 - \delta)^{\lfloor \eta \rfloor}(m - n + m\lfloor n/m \rfloor)/m$$

$$= (1 - \delta)^{\lceil \eta \rceil}(\eta - \lfloor \eta \rfloor) + (1 - \delta)^{\lfloor \eta \rfloor}(1 - \eta + \lfloor \eta \rfloor),$$

and it survives the defenses of all r groups that can fire upon it with probability ρ^r because the groups act independently.

The probability our particular attacker fails to destroy target i is then

$$\gamma \doteq 1 - \alpha \rho^r,$$

where, it turns out, $\gamma < 1$ if $\delta < 1$ and/or $D/rA < 1$.

It is certainly *not* true that the a_i events {attacker j fails to destroy target i}, $j = 1, ..., a_i$, are independent. However, if the a_i attackers are cross-targeted (i.e., launched from different places), and if a_i/A is small, then, to a very good approximation, these a_i events may be treated as independent. This is akin to approximating sampling without replacement by sampling with replacement.

Using the approximation just described, we conclude that target i survives with probability γ^{a_i}. The total expected survival value for attack a is then $\Gamma(a) = \sum_i v_i \gamma^{a_i}$, so that the attacker's problem against a defense with partial coordination is formulated as

$$\mathcal{P}_3 \qquad \text{Minimize } \Gamma(a) = \sum_{i=1}^{T} v_i \gamma^{a_i}$$

$$\text{subject to}$$

$$\sum_{i=1}^{T} a_i \leq A$$

$$a_i \in \mathcal{N}, \quad i = 1, ..., T.$$

For $\gamma \in (0, 1)$ the function γ^{a_i} for $a_i \in \mathcal{N}$ is easily shown to be strictly decreasing with strictly increasing increments. This is sufficient to conclude that the NAT holds for problem $\mathcal{P}_3$.

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Note that r, rather than N, is the appropriate measure of the defense's partial coordination and that m, and hence r, depend on geometrical considerations. Since γ^{a_i} is a decreasing function of r, larger values of r indicate less defensive coordination; $r = 1$ indicates perfect defensive coordination and yields solutions to $\mathcal{P}_3$ that are almost identical to the corresponding solutions to $\mathcal{P}_2$. The range of meaningful values of r is $1 \leq r \leq D$.

2.3.2 Algorithm for Problem $\mathcal{P}_3$

The principal motivation behind the algorithm for Problem $\mathcal{P}_3$ is that we can take advantage of the exponential property of the survival probability function. The change in expected survival value at target i when the attack changes from a_i to $a_i + \Delta$ is $v_i \, \gamma^{a_i}(\gamma^\Delta - 1)$, $\Delta = 1, 2, \ldots$. The algorithm is of the "greedy" type. That is, the allocation of each successive attacker is made to that target offering the greatest immediate decrease in expected survival value, given the allocation to that point.

Algorithm:

Step 1 (Initial Phase):

$$a_i = 0, \; i=1, \ldots, T;$$

$$\text{determine } T' = \max_{t=1,\ldots,T} \left\{ t \mid \sum_{i=1}^{t-1} \lfloor 1 + \ell n(v_t/v_i)/\ell n\gamma \rfloor \leq A \right\};$$

$$a_i = \lfloor 1 + \ell n(v_t/v_i)/\ell n\gamma \rfloor, \; i = 1, \ldots, T'; \quad A' = A - \sum_{i=1}^{T'-1} a_i;$$

$$a_{T'} = \min\{1, A'\}; \quad A' := A' - a_{T'};$$

if $A' = 0$, stop (a is optimal for $\mathcal{P}_3$); else go to step 2.

Step 2 (Intermediate Phase):

$$a_i = a_i + \lfloor A'/T' \rfloor, \; i=1, \ldots, T'; \quad A' := A' - T'\lfloor A'/T' \rfloor;$$

if $A' = 0$, stop (a is optimal for $\mathcal{P}_3$); else go to step 3.

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Step 3 (Final Phase):

Let $i_1, ..., i_{T'}$ be a reordering of $1, ..., T'$ such that

if $1 \le m < n \le T'$, then $v_m \gamma^{a_m} \ge v_n \gamma^{a_n}$;

$a_k := a_k + 1, \quad k = 1, ..., i_{A'}$;

stop (a is optimal for $\mathcal{P}_3$).

3. GROUPS OF EQUAL-VALUED TARGETS

If there are relatively few distinct values among the target values $v_1, ..., v_T$, the algorithms of section 2 can be modified to run much faster. The basis for the increase in speed is the following *Quasiuniform Attack Theorem* (QAT).

Theorem (Quasiuniform Attack Theorem)

With an appropriate survival function (including those of problems $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$) and equal-valued targets, there is a nonincreasing optimal attack a^* with the property $0 \le a_1^* - a_T^* \le 1$.

Proofs of the QAT for problems $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by O'Meara (1988). The proof for problem $\mathcal{P}_3$ follows directly from the convexity of the survival probability function γ^{a_i}.

3.1 Modification of the Algorithm for Problem $\mathcal{P}_1$

To make the necessary branching modification to the algorithm described by O'Meara and Soland (1988a, 1988c), we introduce the following notation. Order the G different values among $v_1, ..., v_T$ as

$$v_1 > \cdots > v_g > \cdots > v_G$$

with n_g targets having value v_g, $g = 1, ... , G$. Furthermore, define

$$T_g \doteq \{i \mid v_i = v_g, \ i = 1, ..., T\},$$

$$\underline{A}_g \doteq \sum_{i \in T_g} \underline{a}_i \text{ and } \overline{A}_g \doteq \sum_{i \in T_g} \overline{a}_i.$$

When a given node $S = \{a \mid \underline{a} \preceq a \preceq \overline{a}\}$ is presented for branching, we now define the index

$$g^* = \min\left\{g \mid 0 < \overline{A}_g - \underline{A}_g = \min\left\{\overline{A}_j - \underline{A}_j > 0 \mid j \, \epsilon \, \{1, ..., G\}\right\}\right\}.$$

We then define the limiting aggregate values of S^L as

$$\underline{A}_g^L = \underline{A}_g \ \forall \ g = 1, ..., G \text{ and } \overline{A}_g^L = \begin{cases} \lfloor (\overline{A}_g + \underline{A}_g)/2 \rfloor, & g = g^* \\ \\ \overline{A}_g, & g \neq g^* \end{cases}$$

and the limiting aggregate values of S^R as

$$\overline{A}_g^R = \overline{A}_g \ \forall \ g = 1, ..., G \text{ and } \underline{A}_g^R = \begin{cases} \lfloor (\overline{A}_g + \underline{A}_g)/2 \rfloor + 1, & g = g^* \\ \\ \underline{A}_g, & g \neq g^*. \end{cases}$$

To maintain nonincreasing limiting attacks, we tighten according to the following rules. In succession, for $g = g^*, ..., G$, we tighten

$$\overline{A}_g^L := \min\left\{\overline{A}_g^L, n_g \left\lfloor \frac{\overline{A}_{g-1}}{n_{g-1}} \right\rfloor\right\},$$

and for $g = g^* - 1, ..., 1$, we tighten

$$\underline{A}_g^R := \max\left\{\underline{A}_g^R, n_g \left\lceil \frac{\underline{A}_{g+1}}{n_{g+1}} \right\rceil\right\}.$$

Define $N_1 \doteq 0$ and $N_g \doteq \sum_{i=1}^{g-1} n_i$ for $g = 2, ..., G$. Then with $\overline{i}_g^* \doteq \overline{A}_g - n_g \lfloor \overline{A}_g/n_g \rfloor$, we may now write the limiting attacks of node S^L as

$$\underline{a}^L = \underline{a}$$

and

$$\overline{a}_{N_g + i}^L = \begin{cases} \lceil \overline{A}_g/n_g \rceil, & i \leq \overline{i}_g^* \\ \\ \lfloor \overline{A}_g/n_g \rfloor, & i > \overline{i}_g^* \end{cases} , \quad i = 1, ..., n_g; \ g = 1, ..., G.$$

With $i_g^* \doteq \underline{A}_g - n_g \lfloor \underline{A}_g/n_g \rfloor$, the limiting attacks of node S^R are

$$\overline{a}^R = \overline{a}$$

and

$$\underline{a}^R_{N_g+i} = \left\{ \begin{array}{ll} \lceil \underline{A}_g/n_g \rceil, & i \leq i_g^* \\[2ex] \lfloor \underline{A}_g/n_g \rfloor, & i > i_g^* \end{array} \right. \quad , \; i=1, ..., n_g; \; g=1, ..., G.$$

3.2 Modification of the Algorithm for Problem $\mathcal{P}_2$

In the algorithm of section 2.2.2, each iteration of the allocation step (step k) caused only the attack at the "critical" target i^* to be changed. In the modified algorithm below, a "critical" group is identified with an allocation to all targets within that group. This faster consumption of resources (attackers) reduces the number of time-consuming probability calculations that needs to be made. We use the same group notation as introduced above.

Algorithm for Problem $\mathcal{P}_2$ with Groups (ψ convex):

Step 0 (Initialization):

$$a_i = 0, \; i=1, ..., T; \; A_1 = A;$$

$$k = 1; \text{ go to step } k.$$

Step k (Allocation):

If $A_k = 0$,

stop (a is optimal for problem $\mathcal{P}_2$);

else

$$x_k = \lceil A_k/T \rceil;$$

$$g^* =$$
$$\min\left\{ g \mid v_g\left[\psi\left(a_{N_g+1}+x_k\right) - \psi\left(a_{N_g+1}\right) \right] = \min_{j=1,...,G}\left\{ v_j\left[\psi\left(a_{N_j+1}+x_k\right) - \psi\left(a_{N_j+1}\right) \right] \right\}, \; g = 1, ..., G \right\};$$

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if $n_g \cdot x_k \leq A_k$ then

$$a_{N_g \cdot + i} := a_{N_g \cdot + i} + \Delta_k, \qquad i=1, ..., n_g \cdot ;$$

$$A_{k+1} := A_k - n_g \cdot x_k;$$

$$k := k+1; \qquad \text{go to step } k.$$

else

$$a_{N_g \cdot + i} := a_{N_g \cdot + i} + x_k, \qquad i=1, ..., A_k - n_g \cdot (x_k - 1);$$

$$a_{N_g \cdot + i} := a_{N_g \cdot + i} + x_k - 1, \qquad i=A_k - n_g \cdot (x_k - 1) + 1, ..., n_g \cdot ;$$

stop (a is optimal for $\mathcal{P}_2$).

3.3 Modification of the Algorithm for Problem $\mathcal{P}_3$

When Problem $\mathcal{P}_3$ has groups of equally valued targets, we may modify the algorithm of section 2.3.2 to achieve some computational efficiencies. As in the preceding section, allocation of resources at each step now applies to all targets within a group rather than to a single target. Of course, a group may consist of only one target.

Algorithm for Problem $\mathcal{P}_3$ with Groups:

Step 1 (Initial Phase):

$$a_i = 0, \quad i=1, ..., T;$$

$$\text{determine } G' = \max_{g=1,...,G} \left\{ g \mid \sum_{i=1}^{g-1} n_g \lfloor 1 + \ell n \left(v_g / v_i \right) \big/ \ell n \ \gamma \rfloor \leq A \right\};$$

$$a_{N_i + j} := \lfloor 1 + \ell n \left(v_{G'} / v_i \right) \big/ \ell n \ \gamma \rfloor, \qquad j=1, ..., n_i \ ; \ i=1, ..., G'-1;$$

$$A' = A - \sum_{i=1}^{G'-1} n_i \, a_{N_i + 1};$$

$$a_{N_{G'} + i} = 1, \ i=1, ..., \min\{A', n_{G'}\};$$

$$A' := A' - \min\{A', n_{G'}\};$$

if $A' = 0$, stop (a is optimal for $\mathcal{P}_3$); else go to step 2.

Step 2 (Intermediate Phase):

$$a_{N_i+j} := a_{N_i+j} + \lfloor A' / \sum_{i=1}^{G'} n_i \rfloor \qquad j=1, ..., n_:, \quad i=1, ..., G';$$

$$A' := A' - \left(\sum_{i=1}^{G'} n_i \right) \lfloor A' / \sum_{i=1}^{G'} n_i \rfloor ;$$

if $A' = 0$, stop (a is optimal for $\mathcal{P}_3$); else go to step 3.

Step 3 (Final Phase):

Let $i, ..., i_{G'}$ be a reordering of $1, ..., G'$ such that

if $1 \leq j < k \leq G'$, then $v_j \gamma^{a_{N_j+1}} \geq v_k \gamma^{a_{N_k+1}}$;

determine $\quad g^* = \max_{g=1,...,G'} \left\{ g \mid N_{i_g} \leq A' \right\}$;

$$a_{N_{i_g+h}} = a_{N_{i_g+h}} + 1, \quad h = 1, ..., n_{i_g}, \quad g=1, ..., g^*- 1;$$

$$A' := A' - N_{i_{g*}} ;$$

$$a_{N_{i_{g*}+h}} := a_{N_{i_{g*}+h}} + 1, \quad h=1, ..., A';$$

stop (a is optimal for $\mathcal{P}_3$).

4. COMPUTATIONAL EXPERIENCE

All algorithms, both initial and modified, have been implemented in FORTRAN with no special compiler options invoked. All computations have been performed on the VAX 11/780 of the School of Engineering and Applied Science, The George Washington University.

4.1 Problem $\mathcal{P}_1$

The algorithm for problem $\mathcal{P}_1$ is the most computationally intense of the three in that it involves two levels of optimization (both attacker and defender), each of which involves a branch-and-bound scheme. Accordingly, only problems of modest size have reasonable run times. Table 1,

however, shows that we can consider solving much larger problems in a timely manner when there are groups of equally valued targets. The CPU times shown in the "singly" column are for the basic algorithm in which targets are considered individually. The times shown in the "Groups" column are for the modified algorithm. The problem data are: (T=) 20 targets, (G=) 3 groups (target values: 2@110, 12@60, 6@10), attack and defense reliabilities $\alpha = \delta = 0.9$, and equal numbers of attackers and defenders (A=D).

Table 1. Run Times for Problem $\mathcal{P}_1$

| A(=D) | CPU Time (secs) | | Reduction Factor |
	Singly	Groups	(Singly ÷ Groups)
60	844.03	2.53	333
70	1178.67	3.30	357
80	1637.69	4.10	399
90	1425.23	4.75	300
100	2091.03	8.15	256
Total Time	7176.65	22.83	314

The significant reduction in CPU time is due to the fact that the attacker's branch-and-bound scheme operates in G–space in the modified algorithm rather than in T–space, as in the basic algorithm.

4.2 Problem $\mathcal{P}_2$

For the fully coordinated defense with no impact point prediction, we can solve larger problems since only the attacker has a real decision on allocating resources. In Table 2 the problem data are: (T=) 100 targets, (G=) 3 groups (target values: (10@22, 60@12, 30@2), attack and defense

reliabilities $\alpha = \delta = .99$, and equal numbers of attackers and defenders (A=D).

Table 2. Run Times for Problem $\mathcal{P}_2$

| A(=D) | CPU Time (secs) | | Reduction Factor |
	Singly	Groups	(Singly ÷ Groups)
200	35.42	.15	236
300	112.94	.28	403
400	262.67	.48	547
500	506.78	.82	618
600	866.44	1.30	666
Total Time	1784.25	3.03	588

Here the reduction in run time can be attributed to the much smaller number of survival probability calculations that need to be made. This is a direct result of the Quasiuniform Attack Theorem.

4.3 Problem $\mathcal{P}_3$

The exponential structure of the objective function in the model formulation for Problem $\mathcal{P}_3$ permits solution of problems much larger than either of the others. Notwithstanding the small run times of the basic algorithm, there is a substantial reduction in run time when advantage is taken of equally valued targets. The data for Table 3 are: fixed attack size of (A=) 6000, (T=) 100 targets, (G=) 3 groups (target values: 10@22, 60@12, 30@2), attack and defense reliabilities $\alpha = .40$, $\delta = .80$, and coordination factor (r=) 3.

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Table 3. Run Times for Problem $\mathcal{P}_3$

| D | CPU Time (secs) | | Reduction Factor |
	Singly	Groups	(Singly ÷ Groups)
14,000	.31	.02	15
15,000	.27	.02	13
16,000	.31	.02	15
17,000	.29	.02	14
18,000	.29	.02	14
Total Time	1.47	.10	14

5. CONCLUDING REMARKS

In this paper we have emphasized the min-max modeling and, especially, the solution efficiency of the attacker/defender problem. We have done so for three different cases that model alternate degrees of defender capability with respect to impact point prediciton and coordination. By tailoring the algorithms to the situation in which only a few different target values are present, we are able to solve problems of realistic size at reasonable cost. Multiple computer runs are thus feasible, and so it becomes practical to examine sensitivity and trade-off issues by solving many problem variations.

One such issue we have already begun to examine is the sensitivity of results to the attacker's perception of the defender's capabilities [see O'Meara and Soland (1988d)]. For example, an attack that is optimal against a defender possessing IPP is not, in general, optimal against the same defender lacking IPP. With efficient algorithms for the three cases of defensive capability examined above, it is quite easy to determine the penalties paid by the attacker for misjudging the capability of the defense. The attacker optimization models developed, and the defender optimization model for the case of IPP, can also be used to examine the sensitivity of results to other parameters. For example, the attacker's

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knowledge of α, D, δ and/or r may be imprecise. Similarly, the defender's knowledge of α, δ and/or r may be imprecise. Examination of the effects of such imprecision may be carried out in a straightforward manner. It is also possible to compare, as a function of the true state of affairs and the respective states perceived by attacker and defender, the actual and anticipated expected total survival values.

ACKNOWLEDGMENTS

REFERENCES

[1] Bracken, J., P. S. Brooks and J. E. Falk, "Robust Preallocated Preferential Defense," *Naval Research Logistics*, 34, 1−22 (1987).

[2] Bracken, J., J. E. Falk and A. J. A. Tai, "Robustness of Preallocated Preferential Defense with Assumed Attack Size and Perfect Attacking and Defending Weapons," *Naval Research Logistics*, 34, 23−41 (1987).

[3] Burr, S., J. E. Falk and A. F. Karr, "Integer Prim-Read Solutions to a Class of Target Defense Problems," *Operations Research*, 33, 726−745 (1985).

[4] Eckler, A. R. and S. A. Burr, *Mathematical Models of Target Coverage and Missile Allocation.* Military Operations Research Society, Alexandria, VA (1972).

[5] Falk, J. E., A Max-Min Result, Memorandum for the Record, Institute for Defense Analyses, Alexandria, VA (1985a).

[6]   Falk, J. E., "Minimal Defenses Enforcing Arbitrary Maximum Damage Bounds," *Proceedings of the 1985 IEEE International Conference on Systems, Man and Cybernetics, Tucson, AZ, November* 12−15, 651−654 (1985b).

[7]   Falk, J. E., Private communication (1987).

[8]   Finn, M. V., *The Value of Area Defense Impact Point Prediction in a Two Layer Defense with Perfect Attackers and Defenders*, P-1902, Institute for Defense Analyses, Alexandria, VA (1986).

[9]   Karr, A. F., *Nationwide Defense Against Nuclear Weapons: Properties of Prim-Read Deployments*, P-1395, Institute for Defense Analyses, Alexandria, VA (1981).

[10]   Karr, A. F., *Minimum Cost Mixtures of Point and Area Defenses Yielding Nearly Constant Payoff per Attacking Weapon, I. The Single Area Case*, Draft Paper, Institute for Defense Analyses, Alexandria, VA (1982).

[11]   Lafon, D. and A. Lahrichi, *A Nonlinear Integer Problem*, presented at the EURO/TIMS Joint International Conference, Paris, July 6-8 (1988).

[12]   Matheson, J. D., *Preferential Strategies*, AR 66-2, Analytic Services Inc., Falls Church, VA (1966).

[13]   Matheson, J. D., *Preferential Strategies with Imperfect Weapons*, AR 67-1, Analytic Services Inc., Falls Church, VA (1967).

[14]   McGarvey, D. C., *Mixed-Strategy Solutions to Integer Prim-Read Defense Problems*, P-7302, The RAND Corporation, Santa Monica, CA (1987).

[15]   O'Meara, N. T., *Optimal Strategies for Problems of Simultaneous Attack Against* an Area Defense, D.Sc. Dissertation, George Washington University (1988).

[16]   O'Meara, N. T. and R. M. Soland, *Optimal Strategies for Problems of Simultaneous Attack Against an Area Defense with Perfect Weapons and Impact Point Prediction*, GWU/SEAS/Serial T-524/88, George Washington University, Washington, DC (1988a).

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[17]   O'Meara, N. T. and R. M. Soland, *Optimal Strategies for Problems of Simultaneous Attack Against an Area Defense without Impact Point Prediction*, GWU/SEAS/Serial T-525/88, George Washington University, Washington, DC (1988b).

[18]   O'Meara, N. T. and R. M. Soland, *Optimal Strategies for Problems of Simultaneous Attack Against an Area Defense with Imperfect Weapons and Impact Point Prediction*, GWU/SEAS/Serial T-526/88, George Washington University, Washington, DC (1988c).

[19]   O'Meara, N. T. and R. M. Soland, "Sensitivity of Expected Survival Value to the Attacker's Perception of Defense Capabilities," *Proceedings of the 27th IEEE Conference on Decision and Control*, 2437−2441, Austin, TX, December 7-9 (1988d).

[20]   Soland, R. M., "Minimum-Cost Mixtures of Area and Point Defenses Assuming Simultaneous Attack," *Naval Research Logistics*, 34, 337−363 (1987a).

[21]   Soland, R. M., "Optimal Terminal Defense Tactics When Several Sequential Engagements are Possible," *Operations Research*, 35, 537−542 (1987b).